Scattering matrix theory for stochastic scalar fields

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We consider scattering of stochastic scalar fields on deterministic as well as on random media, occupying a finite domain. The scattering is characterized by a generalized scattering matrix which transforms the angular correlation function of the incident field into the angular correlation function of the scattered field. Within the accuracy of the first Born approximation this matrix can be expressed in a simple manner in terms of the scattering potential of the scatterer. Apart from determining the angular distribution of the spectral intensity of the scattered field, the scattering matrix makes it possible also to determine the changes in the state of coherence of the field produced on scattering.

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I. INTRODUCTION

The concept of scattering matrix, often called the S matrix, was introduced by Wheeler [1] in 1937 in the context of nuclear physics and independently, somewhat later, by Heisenberg [2] in papers published in the period 1943–1945, in connection with the theory of elementary particles. Since these pioneering investigations have been carried out, scattering matrices were introduced in acoustics by Gerjuoy and Saxon [3] and in electromagnetic theory by van Kampen [4] and Saxon [5]. All these investigations express, in a compact form, the transformation of the wave incident on a physical system into the scattered wave. In recent years many other papers have been published which make use of the scattering matrix for analysis of a variety of problems.

In its traditional formulation, the scattering matrix characterizes the transformation of an incident wave into the total (i.e., incident+scattered) wave, each being represented in the form of a multipole expansion. Specifically, the S matrix indicates how any particular term in the multipole expansion of the converging incident wave is transformed into a particular component of the multipole expansion of the diverging scattered wave, both considered at asymptotically large distances from the scatterer.

In many investigations, particularly in electromagnetic theory and in optics, it is very useful to represent the radiated, the diffracted, and the scattered fields in a complete set of plane-wave modes rather then in a complete set of spherical ones, forming the so-called angular plane-wave spectrum (see Ref. [6], Sec. 3.2). The plane wave modes are of two kinds—the familiar homogeneous waves and the somewhat less familiar inhomogeneous or evanescent waves which decay exponentially in amplitude with increasing distance from sources and from scatterers. The evanescent waves are generated, for example, in total internal reflection. Waves of this type have in recent years become of special interest in the rapidly developing subject of near-field optics [7]. Their effects are not easy to analyze by the use of the multipole expansion.

In this paper we reformulate the S matrix theory in the framework of the angular spectrum representation of plane waves [8] and we illustrate the analysis by considering scattering of a stochastic scalar wave of any state of coherence on deterministic as well as on random media.

II. THEORY

Let us first consider a monochromatic scalar field $U^{(i)}(\mathbf{r};\omega)e^{-i\omega t}$, propagating into the half space $z>0$. We may represent it in the form of an angular spectrum of plane waves (Ref. [6], Sec. 3.2.2), viz.

$$U^{(i)}(\mathbf{r};\omega) = \int a^{(i)}(\mathbf{u};\omega)e^{ik(\mathbf{u}_\perp\cdot\mathbf{r}+u_z z)}d^2\mathbf{u}_\perp, \tag{1}$$

where $k=\omega/c$ is the wave number, c being the speed of light in vacuum; $\mathbf{u}=(u_x,u_y,u_z)$ is a unit vector, $\mathbf{u}_\perp=(u_x,u_y,0)$, and

$$u_z = \sqrt{1-\mathbf{u}_\perp}, \quad \text{when } \mathbf{u}_\perp \leqslant 1, \tag{2a}$$

$$= i\sqrt{\mathbf{u}_\perp - 1}, \quad \text{when } \mathbf{u}_\perp > 1. \tag{2b}$$

The integration in Eq. (1) extends over the whole u_x, u_y plane. Waves for which Eq. (2a) holds represent ordinary *homogeneous* plane waves, those for which Eq. (2b) holds represent *evanescent* plane waves, which decay exponentially in amplitude with increasing distance z from the origin.

Suppose now that the field is incident on a medium, occupying a finite domain D, located in the strip $0\leqslant z\leqslant L$ (see Fig. 1). The scattered field in each of the two half spaces $z<0$ and $z>L$ on either side of the scatterer may also be represented in the form of the angular spectrum of plane waves, i.e., in the form

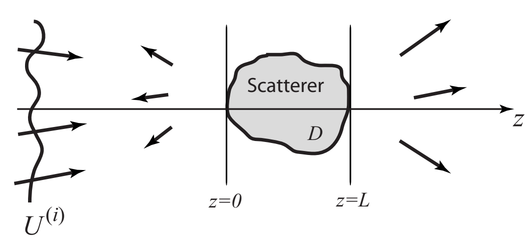

FIG. 1. Illustrating the notation.

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$$U_{\pm}^{(s)}(\mathbf{r};\omega)=\int a^{(s)}(\mathbf{u};\omega)e^{ik(\mathbf{u}_{\perp}\cdot\mathbf{r}\pm u_z z)}d^2\mathbf{u}_{\perp}, \qquad (3)$$

where the positive or negative signs in the exponent taken according as the field point $\mathbf{r}=(x,y,z)$ is located in the half space $z>L$ or $z<0$, respectively. Contributions of evanescent waves will only be significant at points $\mathbf{r}$ which are within a distance on the order of a wavelength or less from the scatterer.

The total field $U^{(t)}$ is the sum of the incident field $U^{(i)}$ and the scattered field $U^{(s)}$,

$$U^{(t)}(\mathbf{r};\omega)=U^{(i)}(\mathbf{r};\omega)+U_{\pm}^{(s)}(\mathbf{r};\omega). \qquad (4)$$

If the incident field is a plane wave the field in the far zone is just the field $U^{(s)}$ scattered in all directions, except in the direction of the incident wave. In the more general case treated in the present paper, the incident field consists of superposition of plane waves falling on the scatterer from many different directions. Under these circumstances the incident field contributes to the total field in the far zone in different directions, not just in a single direction, and the incident and the scattered fields cannot be separated.

We may represent the total field $U^{(t)}(\mathbf{r};\omega)$ also in the form of an angular spectrum of plane waves, viz.

$$U_{\pm}^{(t)}(\mathbf{r};\omega)=\int a^{(t)}(\mathbf{u};\omega)e^{ik(\mathbf{u}_{\perp}\cdot\mathbf{r}\pm u_z z)}d^2\mathbf{u}_{\perp}. \qquad (5)$$

Suppose first that a monochromatic plane wave of frequency ω, with (generally complex) amplitude $a^{(i)}$ is incident on a linear deterministic medium, in a direction specified by a unit vector $\mathbf{u}'$. Then the amplitude of the total scattered field $a^{(t)}(\mathbf{u},\omega)$ in a direction $\mathbf{u}$ can be expressed as

$$a^{(t)}(\mathbf{u},\omega)=\mathbb{S}(\mathbf{u},\mathbf{u}',\omega)a^{(i)}(\mathbf{u}',\omega), \qquad (6)$$

where $\mathbb{S}(\mathbf{u},\mathbf{u}',\omega)$ may be identified with the *(spectral) scattering matrix*. Here, of course, $a^{(t)}$ also depends on $\mathbf{u}'$ (not shown). In the more general case which we are considering here, the incident field consists of many monochromatic plane waves, propagating in different directions and one then has, in place of Eq. (6), the more general relation

$$a^{(t)}(\mathbf{u},\omega)=\int \mathbb{S}(\mathbf{u},\mathbf{u}',\omega)a^{(i)}(\mathbf{u}',\omega)d^2\mathbf{u}'_{\perp}. \qquad (7)$$

It follows from Eqs. (4), (5), and (7) that

$$a^{(t)}(\mathbf{u},\omega)=a^{(i)}(\mathbf{u},\omega)+a^{(s)}(\mathbf{u},\omega). \qquad (8)$$

In view of Eq. (6) and (8), the amplitude function $a^{(i)}(\mathbf{u},\omega)$ of the incident field and the amplitude function $a^{(s)}(\mathbf{u})$ of the scattered (total–incident) field are related by the formula

$$a^{(s)}(\mathbf{u},\omega)=[\mathbb{S}(\mathbf{u},\mathbf{u}',\omega)-1]\mathbf{a}^{(i)}(\mathbf{u}',\omega). \qquad (9)$$

When the incident field $U^{(i)}(\mathbf{r},\omega)$ is represented in a form of an angular spectrum of plane waves, the scattering amplitude $a^{(s)}(\mathbf{u},\omega)$ generated on scattering will be given by the formula

$$a^{(s)}(\mathbf{u},\omega)=\int [\mathbb{S}(\mathbf{u},\mathbf{u}',\omega)-1]a^{(i)}(\mathbf{u}',\omega)d^2\mathbf{u}'_{\perp}. \qquad (10)$$

On substituting from Eq. (7) for $a^{(t)}(\mathbf{u},\omega)$ into Eq. (5) we obtain the following expression for the (total) field generated on scattering:

$$U^{(t)}(\mathbf{r},\omega)=\int\int \mathbb{S}(\mathbf{u},\mathbf{u}',\omega)a^{(i)}(\mathbf{u}',\omega) \times e^{ik(\mathbf{u}_{\perp}\cdot\mathbf{r}\pm u_z z)}d^2\mathbf{u}'_{\perp}d^2\mathbf{u}_{\perp}. \qquad (11)$$

Similarly, on substituting from Eq. (10) into Eq. (3) we obtain the following expression for the scattered field:

$$U^{(s)}(\mathbf{r},\omega)=\int\int [\mathbb{S}(\mathbf{u},\mathbf{u}',\omega)-1]a^{(i)}(\mathbf{u}',\omega) \times e^{ik(\mathbf{u}_{\perp}\cdot\mathbf{r}\pm u_z z)}d^2\mathbf{u}'_{\perp}d^2\mathbf{u}_{\perp}. \qquad (12)$$

Until now we have considered the situation when the incident field is a monochromatic plane wave. Let us now consider scattering of a stochastic scalar wave field. The second-order correlation properties of such a field at a pair of points $\mathbf{r}_1$ and $\mathbf{r}_2$ may be described by its cross-spectral density function (Ref. [6], Sec. 4.7.1),

$$\begin{aligned}W^{(t)}(\mathbf{r}_1,\mathbf{r}_2;\omega)&=\langle U^{(t)*}(\mathbf{r}_1;\omega)U^{(t)}(\mathbf{r}_2;\omega)\rangle\\&=W^{(ii)}(\mathbf{r}_1,\mathbf{r}_2;\omega)+W^{(ss)}(\mathbf{r}_1,\mathbf{r}_2;\omega)\\&\quad+W^{(is)}(\mathbf{r}_1,\mathbf{r}_2;\omega)+W^{(si)}(\mathbf{r}_1,\mathbf{r}_2;\omega), \qquad (13)\end{aligned}$$

where

$$W^{(\alpha\beta)}(\mathbf{r}_1,\mathbf{r}_2;\omega)=\langle U^{(\alpha)*}(\mathbf{r}_1;\omega)U^{(\beta)}(\mathbf{r}_2;\omega)\rangle, \quad (\alpha=i,s;\beta=i,s), \qquad (14)$$

(asterisk denoting the complex conjugate) are the "self-correlation" functions (when $\alpha=\beta$) and cross-correlation functions (when $\alpha\neq\beta$), respectively, of the incident wave and of the scattered wave. The angular brackets denote the statistical average taken over the ensemble of monochromatic realizations of the field, in the sense of coherence theory in the space-frequency domain (Ref. [6], Sec. 4.7.1). On substituting from Eqs. (1) and (3) into Eq. (14) we obtain for the cross-spectral density functions $W^{(\alpha\beta)}$ the formulas

$$W^{(\alpha\beta)}(\mathbf{r}_1,\mathbf{r}_2;\omega)=\int\int \mathcal{A}^{(\alpha\beta)}(\mathbf{u}_1,\mathbf{u}_2;\omega) \times e^{ik(\mathbf{u}_2\cdot\mathbf{r}_2-\mathbf{u}_1\cdot\mathbf{r}_1)}d^2\mathbf{u}_{1\perp}d^2\mathbf{u}_{2\perp} \quad (\alpha=i,s;\beta=i,s), \qquad (15)$$

where $\mathbf{u}_{1\perp}=(u_{1x},u_{1y},0)$, $\mathbf{u}_{2\perp}=(u_{2x},u_{2y},0)$ and

$$\mathcal{A}^{(\alpha\beta)}(\mathbf{u}_1,\mathbf{u}_2;\omega)=\langle a^{(\alpha)*}(\mathbf{u}_1;\omega)a^{(\beta)}(\mathbf{u}_2;\omega)\rangle \quad (\alpha,\beta=i,s) \qquad (16)$$

are the angular correlation functions [10] (see also Ref. [6], Sec. 5.6.3). From now on when $\alpha=\beta$ we will replace the double superscript by a single superscript, i.e., we will write $A^{(i)}$ in place of $A^{(ii)}$, etc.

Let us also express the cross-spectral density function $W^{(t)}(\mathbf{r}_1,\mathbf{r}_2;\omega)$ of the total field in the form of the "double" angular spectrum representation:

$$W^{(t)}(\mathbf{r}_1,\mathbf{r}_2;\omega) = \int\int \mathcal{A}^{(t)}(\mathbf{u}_1,\mathbf{u}_2;\omega) \times e^{ik(\mathbf{u}_2\cdot\mathbf{r}_2-\mathbf{u}_1\cdot\mathbf{r}_1)}d^2\mathbf{u}_{1\perp}d^2\mathbf{u}_{2\perp}. \quad (17)$$

Here the angular correlation function $\mathcal{A}^{(t)}(\mathbf{u}_1,\mathbf{u}_2;\omega)$ is, of course, defined by a formula analogous to Eq. (16), viz.

$$\mathcal{A}^{(t)}(\mathbf{u}_1,\mathbf{u}_2;\omega) = \langle a^{(t)*}(\mathbf{u}_1;\omega)a^{(t)}(\mathbf{u}_2;\omega)\rangle. \quad (18)$$

Let us now relate this function to the angular correlation function $\mathcal{A}^{(i)}(\mathbf{u}_1,\mathbf{u}_2;\omega)$ of the incident field. On substituting from Eq. (6) into Eq. (18) we obtain the formula

$$\mathcal{A}^{(t)}(\mathbf{u}_1,\mathbf{u}_2;\omega) = \mathbb{M}(\mathbf{u}_1,\mathbf{u}_1',\mathbf{u}_2,\mathbf{u}_2';\omega)\mathcal{A}^{(i)}(\mathbf{u}_1',\mathbf{u}_2';\omega), \quad (19)$$

where

$$\mathbb{M}(\mathbf{u}_1,\mathbf{u}_1',\mathbf{u}_2,\mathbf{u}_2';\omega) = \mathbb{S}^*(\mathbf{u}_1,\mathbf{u}_1';\omega)\mathbb{S}(\mathbf{u}_2,\mathbf{u}_2';\omega). \quad (20)$$

We will call the function $\mathbb{M}$ the *pair-scattering matrix*. On substituting from Eq. (19) into Eq. (17) and integrating twice over all the directions contained in the angular spectrum of the incident and of the scattered fields we obtain the following expression for the cross-spectral density function of the total field produced on scattering:

$$W^{(t)}(\mathbf{r}_1,\mathbf{r}_2;\omega) = \int\int\int\int \mathbb{M}(\mathbf{u}_1,\mathbf{u}_2;\mathbf{u}_1',\mathbf{u}_2';\omega)\mathcal{A}^{(i)}(\mathbf{u}_1',\mathbf{u}_2';\omega) \times e^{ik(\mathbf{u}_2\cdot\mathbf{r}_2-\mathbf{u}_1\cdot\mathbf{r}_1)}d^2\mathbf{u}_{1\perp}d^2\mathbf{u}_{2\perp}d^2\mathbf{u}_{1\perp}'d^2\mathbf{u}_{2\perp}'. \quad (21)$$

With the help of the cross-spectral density function $W^{(t)}(\mathbf{r}_1,\mathbf{r}_2;\omega)$ we may at once determine the spectrum $S^{(t)}(\mathbf{r};\omega)$ and also the spectral degree of coherence $\mu^{(t)}(\mathbf{r}_1,\mathbf{r}_2;\omega)$ of the total field using the formulas (see [6], Sec. 4.3.2):

$$S^{(t)}(\mathbf{r};\omega) = W^{(t)}(\mathbf{r},\mathbf{r};\omega) \quad (22)$$

and

$$\mu^{(t)}(\mathbf{r}_1,\mathbf{r}_2;\omega) = \frac{W^{(t)}(\mathbf{r}_1,\mathbf{r}_2;\omega)}{\sqrt{S^{(t)}(\mathbf{r}_1;\omega)}\sqrt{S^{(t)}(\mathbf{r}_2;\omega)}}. \quad (23)$$

III. FAR-ZONE BEHAVIOR

The main formulas which we just derived simplify, of course, when the point $\mathbf{r}$ is in the far zone of the scatterer (Fig. 2). We then have [see Ref. [6], Eq. (3.2.22)]

$$U^{(i)}(r\mathbf{u};\omega) \sim \frac{2\pi i u_z}{k}a^{(i)}(\mathbf{u};\omega)\frac{e^{ikr}}{r}, \quad (24)$$

$$U^{(s)}(r\mathbf{u};\omega) \sim \pm\frac{2\pi i u_z}{k}a^{(s)}(\mathbf{u};\omega)\frac{e^{ikr}}{r}, \quad (25)$$

and

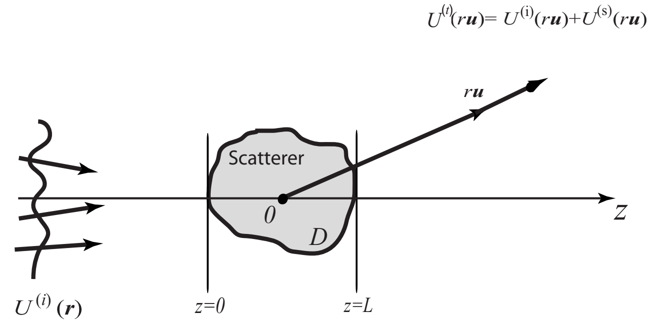

FIG. 2. Notation related to the far-zone field.

$$U^{(t)}(r\mathbf{u};\omega) \sim \pm\frac{2\pi i u_z}{k}a^{(t)}(\mathbf{u};\omega)\frac{e^{ikr}}{r}, \quad (26)$$

as $kr\to\infty$, with the direction $\mathbf{u}$ being kept fixed. In Eqs. (25) and (26) the positive and the negative signs are again taken on the right-hand side according as $u_z > L$ and $u_z < 0$, respectively.

It follows from Eqs. (5), (7), (24), and (26) that the total field $U^{(t)}$ in the far zone may be expressed in the form

$$U^{(t)}(r\mathbf{u};\omega) \sim \pm\frac{2\pi i u_z}{k}\frac{e^{ikr}}{r}\int \mathbb{S}(\mathbf{u},\mathbf{u}';\omega)a^{(i)}(\mathbf{u}';\omega)d^2\mathbf{u}_\perp'. \quad (27)$$

It should be noted that once the (usually complex) amplitudes $a^{(t)}$ of the total scattered waves have been determined from Eq. (7), the total scattered field throughout both half spaces $z > L$ and $z < 0$ can be determined just by substituting the scattering amplitude $a^{(s)}$ and the total amplitude $a^{(t)}$ into the angular spectrum representations.

It follows from Eqs. (13), (16), and (26) that the cross-spectral density function of the total field at a pair of points $r\mathbf{u}_1$ and $r\mathbf{u}_2$ in the far zone at distance r from the scatterer, in directions $\mathbf{u}_1$ and $\mathbf{u}_2$ ($u_1^2 = u_2^2 = 1$) is given by the expression

$$W^{(t)}(r\mathbf{u}_1,r\mathbf{u}_2;\omega) \sim \pm\frac{4\pi^2}{k^2r^2}u_{1z}u_{2z}\mathcal{A}^{(t)}(\mathbf{u}_1,\mathbf{u}_2;\omega), \quad (28)$$

u_{1z}, u_{2z} denoting the z components of the unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively. On substituting from Eq. (19) for the angular correlation function $\mathcal{A}^{(t)}(\mathbf{u}_1,\mathbf{u}_2;\omega)$ into Eq. (28) and integrating over all directions contributing to the incident field we obtain for the cross-spectral density of the total field in the far zone the formula

$$W^{(t)}(r\mathbf{u}_1,r\mathbf{u}_2;\omega) \sim \pm\frac{4\pi^2}{k^2r^2}u_{1z}u_{2z}\int\int \mathbb{M}(\mathbf{u}_1,\mathbf{u}_2;\mathbf{u}_1',\mathbf{u}_2';\omega) \times \mathcal{A}^{(i)}(\mathbf{u}_1',\mathbf{u}_2';\omega)d^2\mathbf{u}_{1\perp}'d^2\mathbf{u}_{2\perp}'. \quad (29)$$

It follows form Eqs. (22), (23), and (28) that the spectrum $S^{(t)}(r\mathbf{u};\omega)$ and the spectral degree of coherence $\mu^{(t)}(r\mathbf{u}_1,r\mathbf{u}_2;\omega)$ of the far field are given by expressions

$$S^{(t)}(r\mathbf{u};\omega) = \frac{4\pi^2u_z^2}{k^2r^2}\mathcal{A}^{(t)}(\mathbf{u},\mathbf{u};\omega) \quad (30)$$

and

$$\mu^{(t)}(r\mathbf{u}_1, r\mathbf{u}_2; \omega) = \frac{\mathcal{A}^{(t)}(\mathbf{u}_1, \mathbf{u}_2, \omega)}{\sqrt{\mathcal{A}^{(t)}(\mathbf{u}_1, \mathbf{u}_1, \omega)}\sqrt{\mathcal{A}^{(t)}(\mathbf{u}_2, \mathbf{u}_2, \omega)}}, \quad (31)$$

respectively, where $\mathcal{A}^{(t)}(\mathbf{u}_1, \mathbf{u}_2; \omega)$ is given by Eq. (19).

IV. RELATION BETWEEN THE SCATTERING MATRIX $\mathbb{M}$ OF THE TOTAL FIELD AND THE SCATTERING POTENTIAL IN THE FIRST BORN APPROXIMATION

Within the accuracy of the first Born approximation the scattering matrix may be expressed in terms of the scattering potential in a simple way. Let $n(\mathbf{r}, \omega)$ be the refractive index distribution throughout the scatterer. The scattering potential $F(\mathbf{r}, \omega)$ is then given by the formula (Ref. [9], p. 696)

$$F(\mathbf{r},\omega) = \frac{k^2}{4\pi}[n^2(\mathbf{r},\omega) - 1], \quad \mathbf{r} \in D$$
$$= 0, \quad \text{otherwise}, \quad (32)$$

where, as before, D denotes the domain occupied by the scatterer and the total scattering amplitude is then given by ([9], p. 700)

$$a^{(t)}(\mathbf{u};\omega) = \tilde{F}[k(\mathbf{u} - \mathbf{u}'), \omega]a^{(i)}(\mathbf{u}';\omega), \quad (33)$$

where

$$\tilde{F}(\mathbf{K},\omega) = \int_D F(\mathbf{r}',\omega)e^{-\mathbf{K}\cdot\mathbf{r}'}d^3\mathbf{r}' \quad (34)$$

is the three-dimensional Fourier transform of the scattering potential. In view of Eqs. (6) and (33), the scattering matrix, within the accuracy of the first Born approximation, is given by the simple expression

$$\mathbb{S}^{(1)}(\mathbf{u},\mathbf{u}';\omega) = \tilde{F}[k(\mathbf{u} - \mathbf{u}'), \omega]. \quad (35)$$

It follows from Eqs. (19) and (35) that

$$\mathcal{A}^{(t)}(\mathbf{u}_1,\mathbf{u}_2;\omega) = \mathbb{M}^{(1)}(\mathbf{u}_1,\mathbf{u}_2;\mathbf{u}_1',\mathbf{u}_2';\omega)\mathcal{A}^{(i)}(\mathbf{u}_1',\mathbf{u}_2';\omega), \quad (36)$$

where

$$\mathbb{M}^{(1)}(\mathbf{u}_1,\mathbf{u}_2;\mathbf{u}_1',\mathbf{u}_2';\omega) = \tilde{F}^*[k(\mathbf{u}_1 - \mathbf{u}_1'), \omega]\tilde{F}[k(\mathbf{u}_2 - \mathbf{u}_2'), \omega] \quad (37)$$

is the pair-scattering matrix of the total field.

V. SCATTERING ON RANDOM MEDIA

We have assumed so far that the scatterer is deterministic. It is not difficult to generalize our analysis to scattering on a random medium. It follows from Eq. (20) that the second-order correlation properties of such a medium may be characterized in terms of the "averaged pair-scattering matrix"

$$\mathbb{M}_m(\mathbf{u}_1,\mathbf{u}_1',\mathbf{u}_2,\mathbf{u}_2';\omega) = \langle \mathbb{S}^*(\mathbf{u}_1,\mathbf{u}_1';\omega)\mathbb{S}(\mathbf{u}_2,\mathbf{u}_2';\omega)\rangle_m, \quad (38)$$

where the angular brackets with subscript "m" denote the average, taken over the ensemble of realizations of the scattering medium. In view of Eq. (29) the cross-spectral density function of the total far field, generated by scattering from a random medium, is given by the expression

$$W^{(t)}(r\mathbf{u}_1, r\mathbf{u}_2;\omega) = \frac{4\pi^2}{k^2r^2}u_{1z}u_{2z}\int\int \mathbb{M}_m(\mathbf{u}_1,\mathbf{u}_2;\mathbf{u}_1',\mathbf{u}_2';\omega) \times \mathcal{A}^{(i)}(\mathbf{u}_1',\mathbf{u}_2';\omega)d^2\mathbf{u}_{1\perp}'d^2\mathbf{u}_{2\perp}'. \quad (39)$$

From Eq. (37) it follows at once on averaging that, within the accuracy of the first Born approximation,

$$\mathbb{M}_m^{(1)}(\mathbf{u}_1,\mathbf{u}_1',\mathbf{u}_2,\mathbf{u}_2';\omega) = \langle \tilde{F}^*[k(\mathbf{u}_1 - \mathbf{u}_1');\omega]\tilde{F}[k(\mathbf{u}_2 - \mathbf{u}_2');\omega]\rangle_m. \quad (40)$$

This function can also be expressed in terms of the correlation function

$$C_F(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle F^*(\mathbf{r}_1,\omega)F(\mathbf{r}_2,\omega)\rangle_m \quad (41)$$

of the scattering potential (32) as is seen from the following simple argument (cf. Ref. [11]): if $\tilde{C}_F(\mathbf{K}_1, \mathbf{K}_2, \omega)$ denotes the six-dimensional spatial Fourier transform of the correlation function $C_F(\mathbf{r}_1, \mathbf{r}_2, \omega)$, i.e., if

$$\tilde{C}_F(\mathbf{K}_1,\mathbf{K}_2,\omega) = \int\int C_F(\mathbf{r}_1,\mathbf{r}_2,\omega)e^{-i(\mathbf{K}_1\mathbf{r}_1+\mathbf{K}_2\mathbf{r}_2)}d^3\mathbf{r}_1d^3\mathbf{r}_2, \quad (42)$$

then

$$\mathbb{M}_m^{(1)}(\mathbf{u}_1,\mathbf{u}_1',\mathbf{u}_2,\mathbf{u}_2';\omega) = \tilde{C}_F[-k(\mathbf{u}_1 - \mathbf{u}_1'),k(\mathbf{u}_2 - \mathbf{u}_2');\omega]. \quad (43)$$

The expression (39) for the cross-spectral density function of the far field, valid within the accuracy of the first Born approximation, becomes

$$W^{(t)}(r\mathbf{u}_1, r\mathbf{u}_2;\omega) = \frac{4\pi^2}{k^2r^2}u_{1z}u_{2z} \times\int\int \tilde{C}_F[-k(\mathbf{u}_1 - \mathbf{u}_1'),k(\mathbf{u}_2 - \mathbf{u}_2');\omega] \times \mathcal{A}^{(i)}(\mathbf{u}_1',\mathbf{u}_2';\omega)d^2\mathbf{u}_{1\perp}'d^2\mathbf{u}_{2\perp}' \quad (kr \to \infty). \quad (44)$$

VI. EXAMPLE OF TWO CORRELATED PLANE WAVES SCATTERING ON A DETERMINISTIC MEDIUM

As an example of some of the main formulas that we just derived, we consider an incident field $U^{(i)}$ consisting of two mutually correlated homogeneous plane waves propagating along directions $\mathbf{u}_1'$ and $\mathbf{u}_2'$ (see Fig. 3), scattered by a sphere. The spectral amplitude $a^{(i)}(\mathbf{u}', \omega)$ of the incident field has then the form

$$a^{(i)}(\mathbf{u}',\omega) = a^{(i)}(\mathbf{u}_1',\omega)\delta^{(2)}(\mathbf{u}' - \mathbf{u}_1') + a^{(i)}(\mathbf{u}_2',\omega)\delta^{(2)}(\mathbf{u}' - \mathbf{u}_2'), \quad (45)$$

$\delta^{(2)}(\mathbf{u})$ being the spherical Dirac δ function [12].

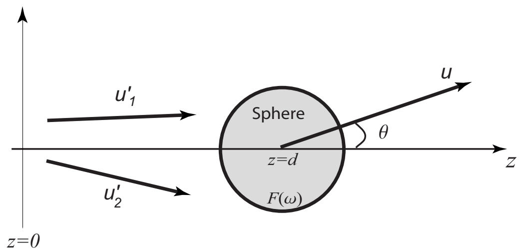

FIG. 3. Notation relating to scattering of two correlated plane waves by a sphere.

On substituting from Eq. (45) into Eq. (16), with $\alpha=\beta=i$, we find that the angular correlation function of the incident field has the form

$$\mathcal{A}^{(i)}(\mathbf{u}_1',\mathbf{u}_2';\omega)=\mathbf{a}(\mathbf{u}_1',\mathbf{u}_1';\omega)\delta^{(2)}(\mathbf{u}'-\mathbf{u}_1')\delta^{(2)}(\mathbf{u}'-\mathbf{u}_1') + \mathbf{a}(\mathbf{u}_2',\mathbf{u}_2';\omega)\delta^{(2)}(\mathbf{u}'-\mathbf{u}_2')\delta^{(2)}(\mathbf{u}'-\mathbf{u}_2') + \mathbf{a}(\mathbf{u}_1',\mathbf{u}_2';\omega)\delta^{(2)}(\mathbf{u}'-\mathbf{u}_1')\delta^{(2)}(\mathbf{u}'-\mathbf{u}_2') + \mathbf{a}(\mathbf{u}_2',\mathbf{u}_1';\omega)\delta^{(2)}(\mathbf{u}'-\mathbf{u}_2')\delta^{(2)}(\mathbf{u}'-\mathbf{u}_1'), \tag{46}$$

where

$$\mathbf{a}(\mathbf{u}_p',\mathbf{u}_q';\omega)=\langle a^{(i)*}(\mathbf{u}_p',\omega)a^{(i)}(\mathbf{u}_q',\omega)\rangle \quad (p,q=1,2), \tag{47}$$

the angular brackets again denoting the average over the ensemble of the incident field.

Let us consider the cross-spectral density $W^{(t)}$ of the total field in the far zone, produced by scattering of the two plane waves. On substituting from Eq. (46) into Eq. (28) and on using the sifting property of the δ function we obtain for $W^{(t)}$ the expression

$$W^{(t)}(r\mathbf{u}_1,r\mathbf{u}_2;\omega)=\frac{4\pi^2u_{1z}u_{2z}}{k^2r^2} \times\{\mathbb{M}(\mathbf{u}_1,\mathbf{u}_2;\mathbf{u}_1',\mathbf{u}_1';\omega)\mathbf{a}(\mathbf{u}_1',\mathbf{u}_1';\omega) + \mathbb{M}(\mathbf{u}_1,\mathbf{u}_2;\mathbf{u}_2',\mathbf{u}_2';\omega)\mathbf{a}(\mathbf{u}_2',\mathbf{u}_2';\omega) + \mathbb{M}(\mathbf{u}_1,\mathbf{u}_2;\mathbf{u}_1',\mathbf{u}_2';\omega)\mathbf{a}(\mathbf{u}_1',\mathbf{u}_2';\omega) + \mathbb{M}(\mathbf{u}_1,\mathbf{u}_2;\mathbf{u}_2',\mathbf{u}_1';\omega)\mathbf{a}(\mathbf{u}_2',\mathbf{u}_1';\omega)\}. \tag{48}$$

Using this expression one can readily determine the spectral density of the total field, given by Eq. (30), in the far zone, along the direction specified by unit vector $\mathbf{u}$:

$$S^{(t)}(r\mathbf{u};\omega)=\frac{4\pi^2u_z^2}{k^2r^2}\{\mathbb{M}(\mathbf{u},\mathbf{u};\mathbf{u}_1',\mathbf{u}_1';\omega)\mathbf{a}(\mathbf{u}_1',\mathbf{u}_1';\omega) + \mathbb{M}(\mathbf{u},\mathbf{u};\mathbf{u}_2',\mathbf{u}_2';\omega)\mathbf{a}(\mathbf{u}_2',\mathbf{u}_2';\omega) + 2\,\mathrm{Re}[\mathbb{M}(\mathbf{u},\mathbf{u};\mathbf{u}_1',\mathbf{u}_2';\omega)\mathbf{a}(\mathbf{u}_1',\mathbf{u}_2';\omega)]\}. \tag{49}$$

Here Re denotes the real part.

Further, we see from Eqs. (48) and (31) that the spectral degree of coherence $\mu^{(t)}$ of the total field in the far zone is given by the formula

$$\mu^{(t)}(r\mathbf{u}_1,r\mathbf{u}_2;\omega)=\frac{1}{\sqrt{S^{(t)}(r\mathbf{u}_1;\omega)}\sqrt{S^{(t)}(r\mathbf{u}_2;\omega)}} \times\{\mathbb{M}^{(1)}(\mathbf{u}_1,\mathbf{u}_2;\mathbf{u}_1',\mathbf{u}_1';\omega)\mathbf{a}(\mathbf{u}_1',\mathbf{u}_1';\omega) + \mathbb{M}^{(1)}(\mathbf{u}_1,\mathbf{u}_2;\mathbf{u}_2',\mathbf{u}_2';\omega)\mathbf{a}(\mathbf{u}_2',\mathbf{u}_2';\omega) + \mathbb{M}^{(1)}(\mathbf{u}_1,\mathbf{u}_2;\mathbf{u}_1',\mathbf{u}_2';\omega)\mathbf{a}(\mathbf{u}_1',\mathbf{u}_2';\omega) + \mathbb{M}^{(1)}(\mathbf{u}_1,\mathbf{u}_2;\mathbf{u}_2',\mathbf{u}_1';\omega)\mathbf{a}(\mathbf{u}_2',\mathbf{u}_1';\omega)\}. \tag{50}$$

Suppose that the angular correlation function $\mathbf{a}(\mathbf{u}_1',\mathbf{u}_2';\omega)$ of the incident field has Gaussian form, i.e. that

$$\mathbf{a}(\mathbf{u}_p',\mathbf{u}_q';\omega)=\mathfrak{a}_{pq}e^{-(k^2\Delta^2/2)(\mathbf{u}_q'-\mathbf{u}_p')^2} \quad (p,q=1,2), \tag{51}$$

where $\mathfrak{a}_{pq}$ and Δ depend, in general, on the frequency ω.

Suppose that the scatterer is a sphere centered at a point $\mathbf{r}_c=(0,0,d)$, with a three-dimensional (soft) Gaussian potential

$$F(\mathbf{r};\omega)=B\exp\left[-\frac{x^2+y^2+(z-d)^2}{2\sigma^2}\right]. \tag{52}$$

The variance σ^2 is taken to be independent of position but, in general, will dependent on the frequency. The three-dimensional Fourier transform, defined by Eq. (34) of the expression Eq. (52), is

$$\tilde{F}(\mathbf{K};\omega)=B(2\pi)^{(3/2)}\sigma^3e^{-K^2\sigma^2/2}e^{idK_z}. \tag{53}$$

On substituting from Eq. (53), into Eq. (37) and setting $\mathbf{K}=k(\mathbf{u}-\mathbf{u}')$, we find that, within the accuracy of the first Born approximation, the pair scattering matrix is

$$\mathbb{M}^{(1)}(\mathbf{u}_1,\mathbf{u}_2;\mathbf{u}_1',\mathbf{u}_2';\omega)=B(2\pi)^3\sigma^6e^{-(k^2\sigma^2/2)(\mathbf{u}_1-\mathbf{u}_1')^2} \times e^{-(k^2\sigma^2/2)(\mathbf{u}_2-\mathbf{u}_2')^2} \times e^{ikd(u_{1z}-u_{1z}')}e^{ikd(u_{2z}-u_{2z}')}. \tag{54}$$

Next we substitute from Eqs. (51) and (54) into Eq. (49). We then readily obtain for the spectral density of the far field the expression

$$S^{(t)}(r\mathbf{u};\omega)=\frac{B(2\pi)^5\sigma^6u_z^2}{k^2r^2}\Big[e^{-k^2\sigma^2(\mathbf{u}-\mathbf{u}_1')^2}\mathfrak{a}_{11}+e^{-k^2\sigma^2(\mathbf{u}-\mathbf{u}_2')^2}\mathfrak{a}_{22} + 2e^{-(k^2\sigma^2/2)(\mathbf{u}-\mathbf{u}_1')^2}e^{-(k^2\sigma^2/2)(\mathbf{u}-\mathbf{u}_2')^2} \times e^{-(k^2\Delta^2/2)(\mathbf{u}_2'-\mathbf{u}_1')^2}\,\mathrm{Re}\{\mathfrak{a}_{12}e^{ikd(u_{2z}'-u_{1z}')}\}\Big]. \tag{55}$$

Figures 4(a)–4(c) show the behavior of the normalized spectral density of the far-field obtained by scattering of two plane waves, for three selected values of parameter $k\Delta$,

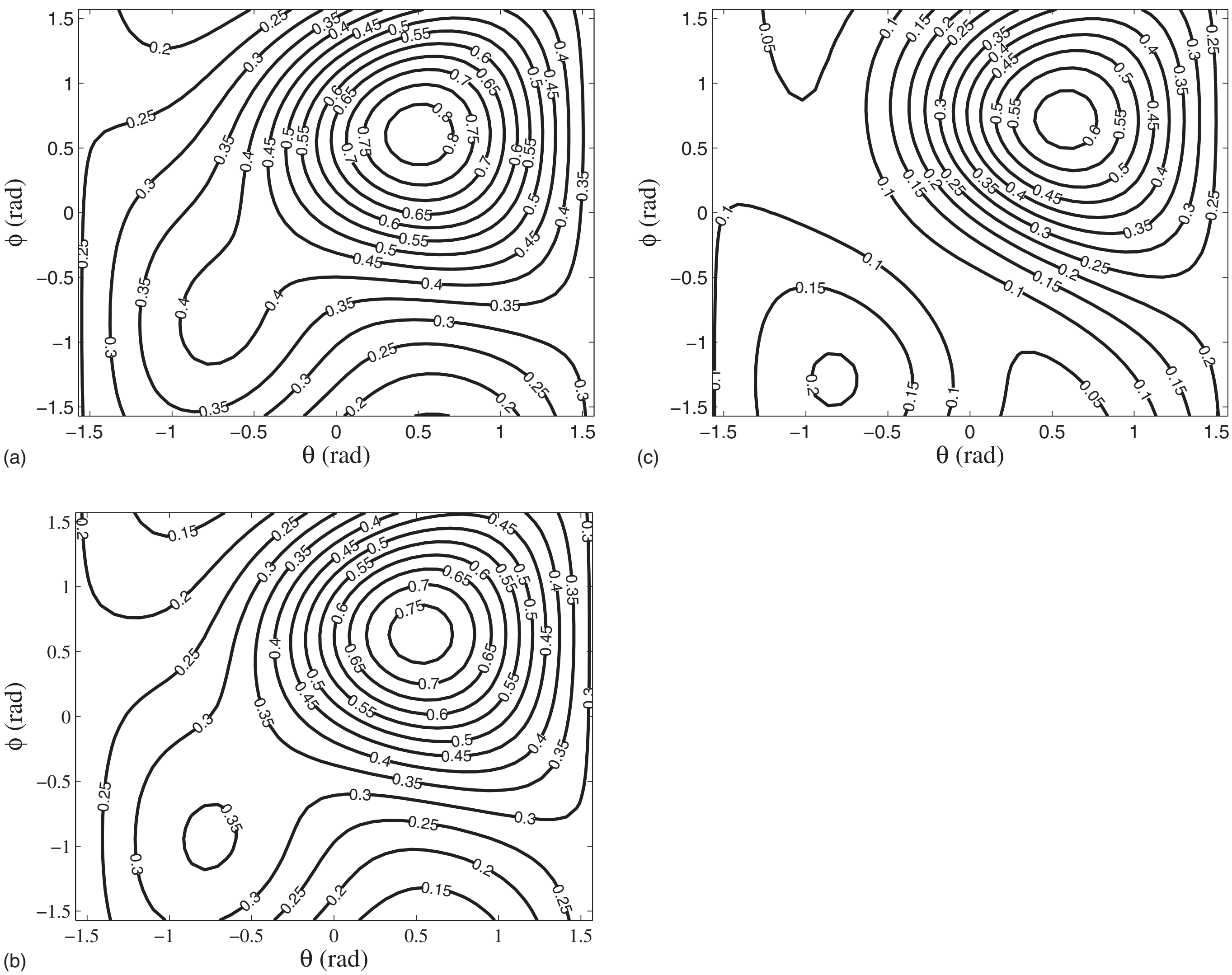

FIG. 4. Contours of the spectral density of the far field, normalized by the spectral density of the incident field, produced by scattering of two correlated plane waves on a sphere with Gaussian potential. The parameters were taken as: $\lambda=0.633\times10^{-6}$ m, $A=1$, $B=1$, $d=1$, $a_1=0.6e^{i\pi/7}$, $a_2=0.9e^{i\pi/6}$, $\theta_1=-\pi/4$; $\phi_1=-\pi/3$; $\theta_2=\pi/6$; $\phi_2=\pi/5$; $k\sigma=1$, (a) $k\Delta=10$ (highly correlated); (b) $k\Delta=1$ (partially correlated); (c) $k\Delta=0.1$ (nearly uncorrelated).

calculated from Eq. (55). The angles θ and ϕ are the polar and the azimuthal angles of the unit vector $\mathbf{u}$ in spherical coordinates, i.e., $u_x=\cos\theta\cos\phi$, $u_y=\cos\theta\sin\phi$, $u_z=\sin\theta$.

On substituting from Eqs. (51) and (54) into the expression (50) we obtain for the degree of coherence the formula

$$\mu^{(t)}(r\mathbf{u}_1,r\mathbf{u}_2;\omega)=\frac{e^{ikd(u_{2z}-u_{1z})}}{\sqrt{S^{(t)}(r\mathbf{u}_1;\omega)}\sqrt{S^{(t)}(r\mathbf{u}_2;\omega)}}\{e^{-(k^2\sigma^2/2)(\mathbf{u}_1-\mathbf{u}_1')^2}e^{-(k^2\sigma^2/2)(\mathbf{u}_2-\mathbf{u}_1')^2}\mathfrak{a}_{11}+e^{-(k^2\sigma^2/2)(\mathbf{u}_1-\mathbf{u}_2')^2}e^{-(k^2\sigma^2/2)(\mathbf{u}_2-\mathbf{u}_2')^2}\mathfrak{a}_{22}$$
$$+[e^{-(k^2\sigma^2/2)(\mathbf{u}_1-\mathbf{u}_1')^2}e^{-(k^2\sigma^2/2)(\mathbf{u}_2-\mathbf{u}_2')^2}\mathfrak{a}_{12}e^{ikd(u_{1z}'-u_{2z}')}+e^{-(k^2\sigma^2/2)(\mathbf{u}_1-\mathbf{u}_2')^2}e^{-(k^2\sigma^2/2)(\mathbf{u}_2-\mathbf{u}_1')^2}\mathfrak{a}_{21}e^{ikd(u_{2z}'-u_{1z}')}]$$
$$\times e^{-(k^2\Delta^2/2)(\mathbf{u}_1'-\mathbf{u}_2')^2}\}. \qquad (56)$$

Figures 5 and 6 show the behavior of the modulus of the spectral degree of coherence $\mu^{(t)}(r\mathbf{u}_1,r\mathbf{u}_2;\omega)$ of the far field obtained by scattering of two correlated plane waves on the sphere with Gaussian potential, calculated from Eq. (56). The plane waves are incident on the sphere along directions specified by polar angles $\phi_1'=\pi/2$, $\phi_2'=-\pi/2$ and azimuthal

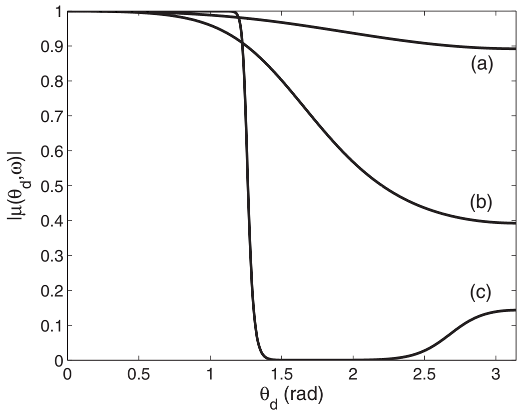

FIG. 5. Modulus of the spectral degree of coherence of the far field, produced by scattering of two correlated plane waves on spheres of different sizes, with Gaussian potential. The parameters were taken as: $\lambda=0.633$ μm, $A=1$, $d=1$, $a_1=1$, $a_2=1$, $k\sigma=1$, $k\Delta=1$ (a) $k\sigma=0.5$ (small sphere); (b) $k\sigma=1$ (sphere with radius on the order of the wavelength); (c) $k\sigma=5$ (large sphere).

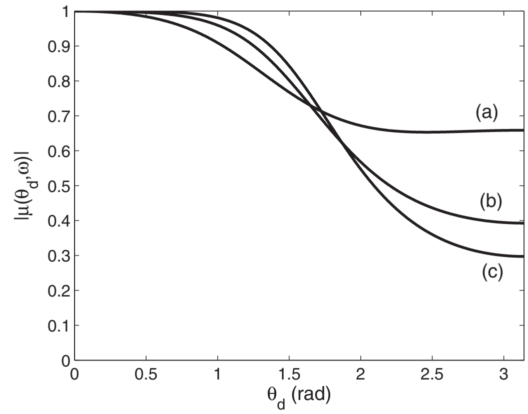

FIG. 6. Modulus of the spectral degree of coherence of the far field, produced by scattering of two plane waves with different degrees of correlation on a sphere with Gaussian potential. The parameters were taken as: $\lambda=0.633$ μm, $A=1$, $d=1$, $a_1=1$, $a_2=1$, $k\sigma=1$, (a) $k\Delta=0.1$ (highly correlated); (b) $k\Delta=1$ (partially correlated); (c) $k\Delta=10$ (nearly uncorrelated).

angles $\theta_1'=\theta_2'=0.1$ rad. The modulus of the degree of coherence of the scattered field was calculated as a function of the angle $\theta_d=\theta_2$, while the other angles were kept fixed: $\theta=0$, $\phi_1=\pi/2$, $\phi_2=-\pi/2$.

Figure 5 shows the behavior of $|\mu^{(t)}(r\mathbf{u}_1,r\mathbf{u}_2;\omega)|$ for three selected values of the scaled size $k\sigma$ of the sphere [curves 5(a)–5(c)], while the relative degree of correlation $k\Delta$ of the incident plane waves is kept fixed.

Figure 6 shows the behavior of $|\mu^{(t)}(r\mathbf{u}_1,r\mathbf{u}_2;\omega)|$ for three selected values of the parameter $k\Delta$ [curves 6(a)–6(c)], while the scaled size of the sphere $k\sigma$ is kept fixed.

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